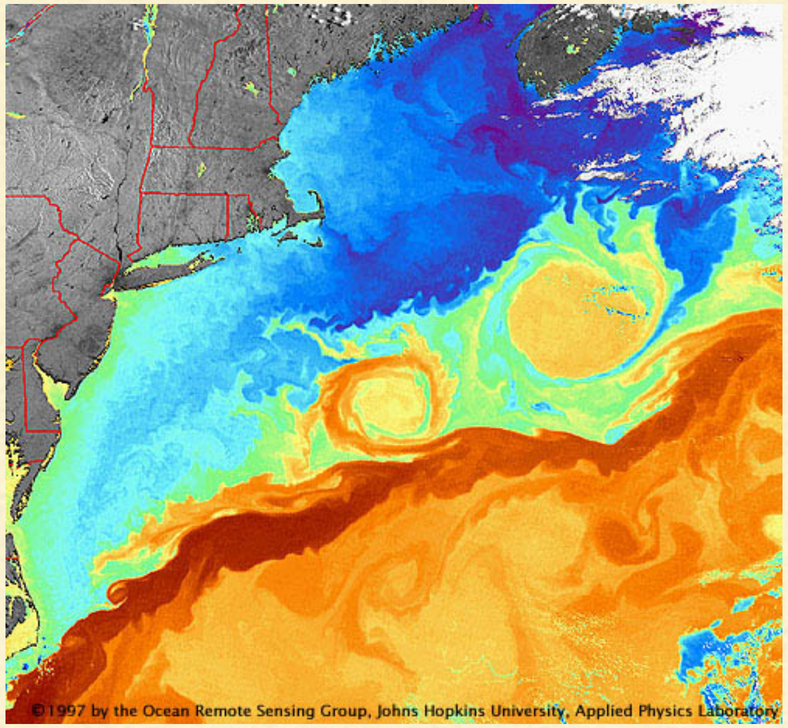
RELEVANCE OF CONSERVATION LAWS FOR AN ENSEMBLE KALMAN FILTER

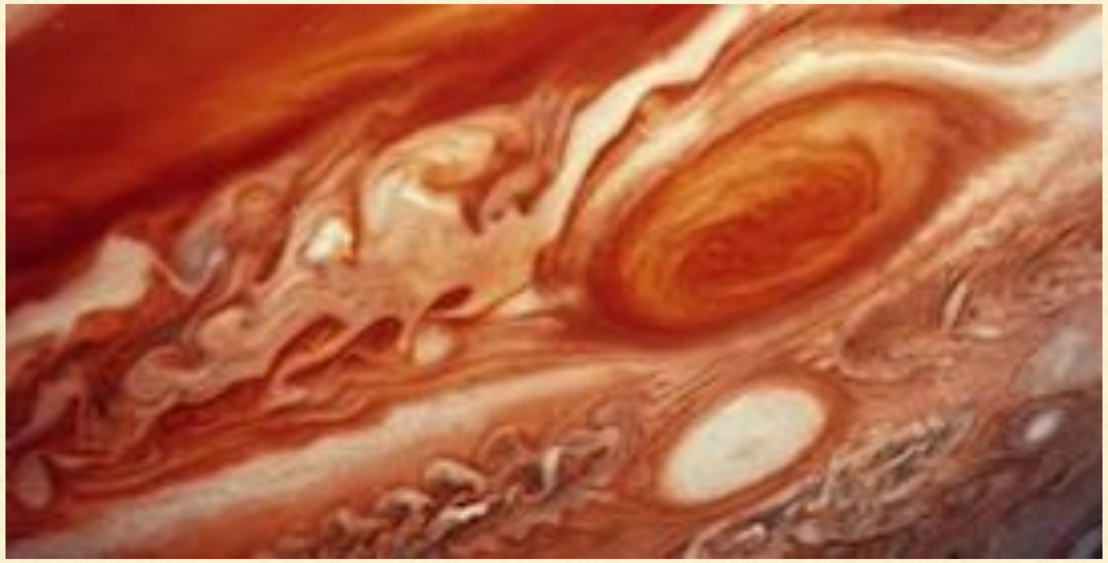
Svetlana Dubinkina (CWI, Amsterdam)

LARGE-SCALE STRUCTURES

Motivation is to predict behaviour of geophysical flows at large scales



Gulf-stream rings



Jupiter Great Red Spot

STATISTICAL EQUILIBRIUM THEORIES

Statistical equilibrium theories aim at predicting coherent large-scale structures.

In statistical equilibrium theories, such a coherent coarse-grained structure is the most probable macrostate and it is defined by conserved quantities associated with all possible microstates.

How to find those microstates? Using evolution of the dynamics.

ILLUSTRATION OF STATISTICAL EQUILIBRIUM THEORY ON QUASI-GEOSTROPHIC FLOW

$$q_t = q_x \psi_y - q_y \psi_x \qquad \Delta\psi = q - h \qquad (x, y) \in [0, 2\pi) \times [0, 2\pi)$$

q is the potential vorticity, ψ is the stream function,
h is the orography, Δ is the Laplace operator.

Conserved quantities are

Energy $E = -1/2 \int \psi(q-h)dxdy$

Casimirs $C_f = \int f(q)dxdy$

DEFINITIONS

Microstate is $q(x, y)$

Macrostate is defined by the probability density function $\rho(x, y, \sigma)$ of having $q(x, y) = \sigma$

The coarse-grained or macroscopic vorticity is defined as

$$\langle q(x, y) \rangle = \int \sigma \rho d\sigma$$

The most probable macrostate $\rho^*(x, y, \sigma)$ is the maximiser of

$$S = -\int dx dy d\sigma \rho \ln \rho$$

subjected to satisfy conservations laws

DIFFERENT STATISTICAL EQUILIBRIUM THEORIES

The QG model has an infinite number of conserved quantities:

- Energy $E = -1/2 \int \psi(q-h)dxdy$
- Casimirs (with any smooth function) $C_f = \int f(q)dxdy$

When deriving a statistical equilibrium theory, one can only take into account some of these conserved quantities. Thus one needs to make a choice which of these is statistically relevant?

ENERGY STATISTICAL THEORY

Assume that the only statistically relevant conserved quantity is energy

$$E = -1/2 \int \psi(q-h)dxdy$$

Then the most probable macrostate is

$$\rho^* = N^{-1} \exp(-\lambda E)$$

The coarse-grained vorticity and stream function are

$$\langle q \rangle = h \qquad \langle \psi \rangle = 0$$

ENSTROPHY STATISTICAL THEORY

Assume that the only statistically relevant conserved quantity is enstrophy (second order Casimir)

$$Z = 1/2 \int q^2 dx dy$$

Then the most probable macrostate is

$$\rho^* = N^{-1} \exp(-\alpha Z)$$

The coarse-grained vorticity and stream function are

$$\langle q \rangle = 0 \qquad \langle \psi \rangle = -\Delta^{-1} h$$

ENERGY-ENSTROPHY STATISTICAL THEORY

Assume that the only statistically relevant conserved quantity are energy and enstrophy

$$E = -1/2 \int \psi(q-h)dxdy \qquad Z = 1/2 \int q^2 dxdy$$

Then the most probable macrostate is

$$\rho^* = N^{-1} \exp(-\beta(Z + \mu E))$$

The coarse-grained vorticity and stream function are

$$\langle q \rangle = \mu \langle \psi \rangle \qquad (\mu - \Delta)\langle \psi \rangle = h$$

ASSUMPTION OF ERGODICITY

The most probable macrostate $\rho^*(x,y,\sigma)$ is the maximum of

$$S = -\int dx dy d\sigma \, \rho \ln \rho$$

subjected to satisfy conservations laws

The coarse-grained or macroscopic vorticity is

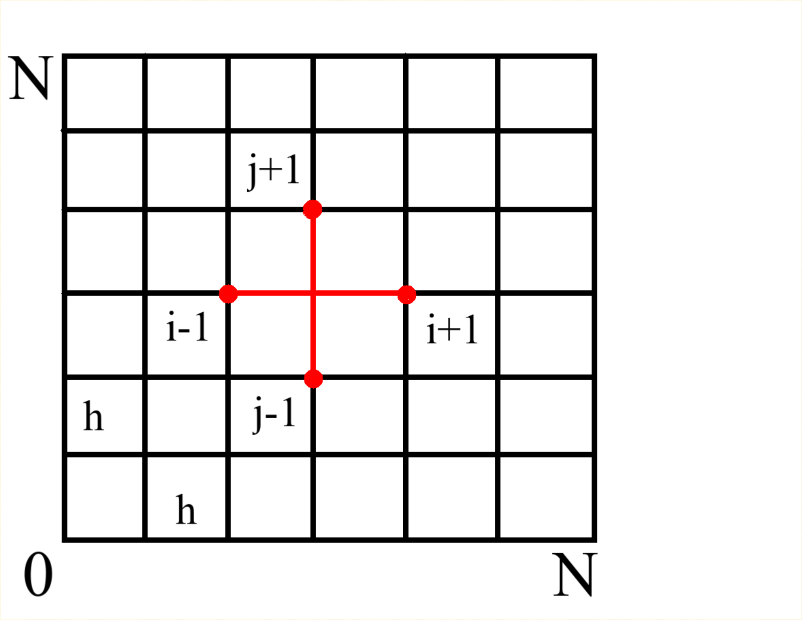
$$\langle q(x,y)\rangle = \int \sigma \rho^*(x,y,\sigma) d\sigma$$

Assumption of ergodicity is that

$$\langle q(x,y)\rangle = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} q(x,y,t) dt$$

subjected to conservative dynamical evolution of q

ARAKAWA DISCRETIZATIONS



The QG model $q_t = q_x \psi_y - q_y \psi_x$

Arakawa discretizations are classical finite difference schemes based on the following equivalent formulation of the right hand side

$$q_x \psi_y - q_y \psi_x \equiv (q\psi_y)_x - (q\psi_x)_y \equiv (\psi q_x)_y - (\psi q_y)_x$$

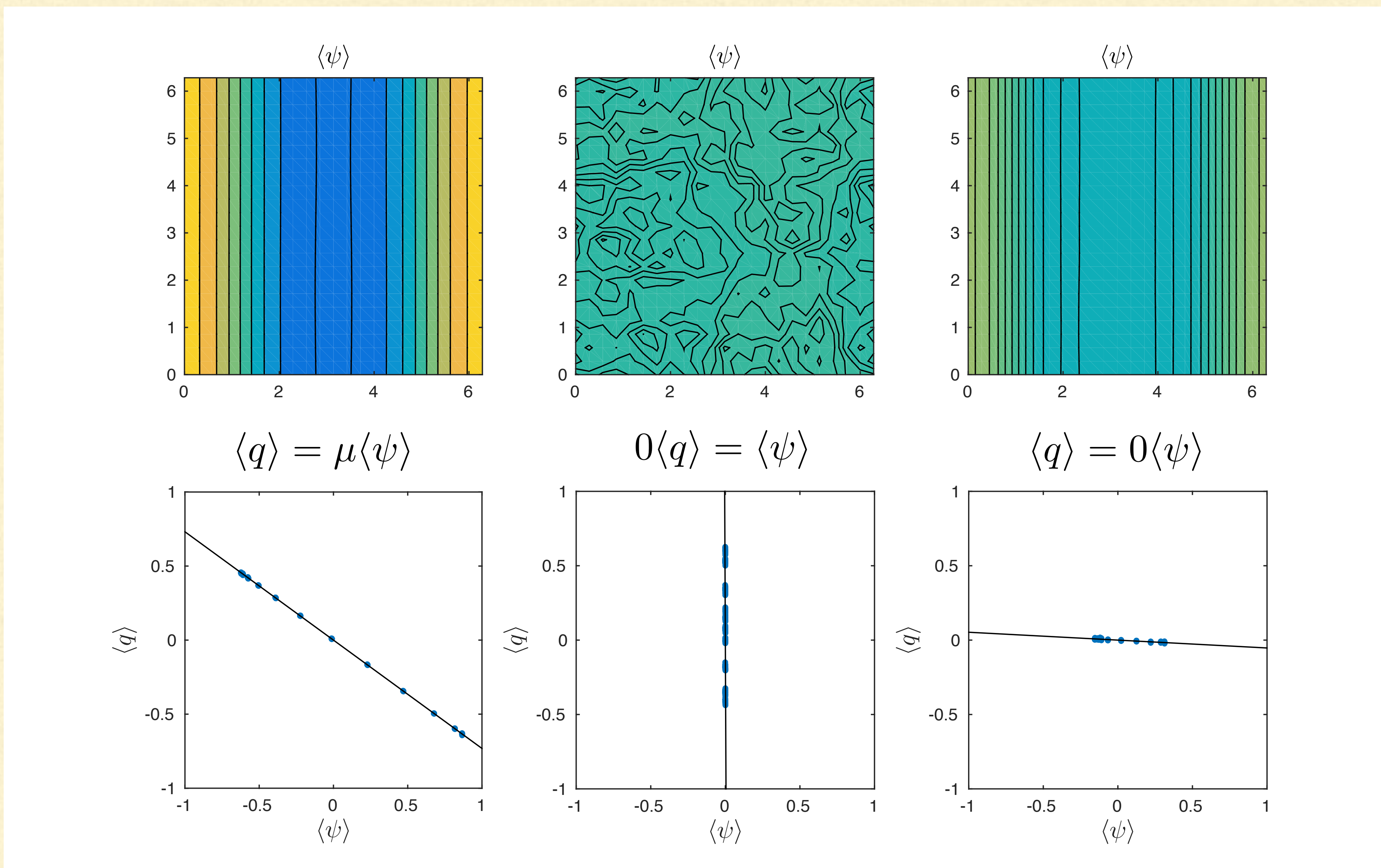
Arakawa E discretisation (preserves energy E)

Arakawa Z discretisation (preserves enstrophy Z)

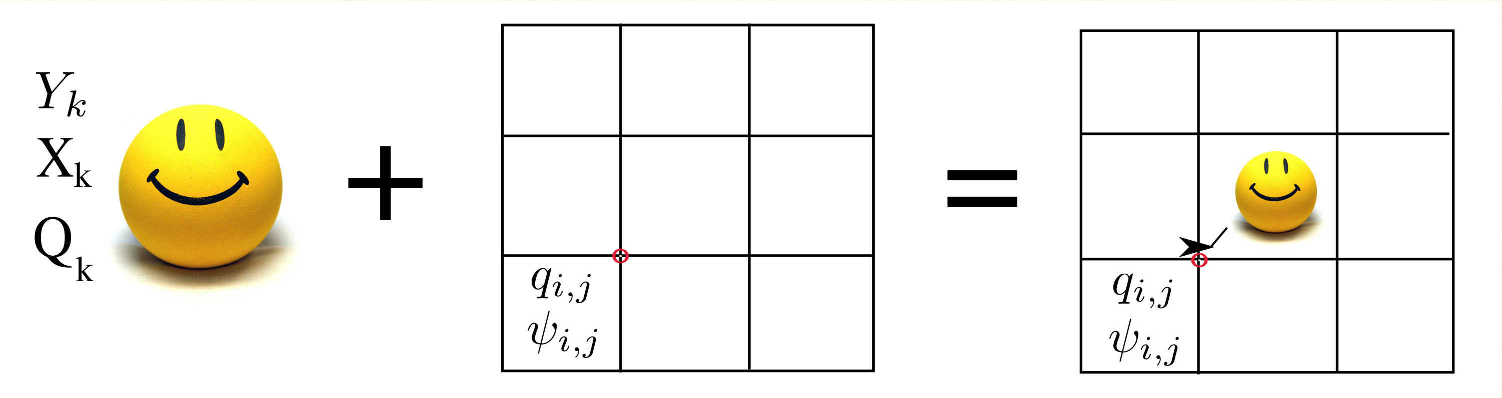
Arakawa EZ discretisation (preserves energy E and enstrophy Z)

COARSE-GRAINED FIELDS OBTAINED BY ARAKAWA DISCRETIZATIONS

Arakawa EZ	Arakawa E	Arakawa Z
$(\mu - \Delta)\langle\psi\rangle = h$	$\langle\psi\rangle = 0$	$\langle\psi\rangle = -\Delta^{-1}h$



HAMILTONIAN PARTICLE-MESH METHOD (EULERIAN-LAGRANGIAN METHOD)



$$q_t = q_x \psi_y - q_y \psi_x$$

$$\Delta \psi = q - h$$

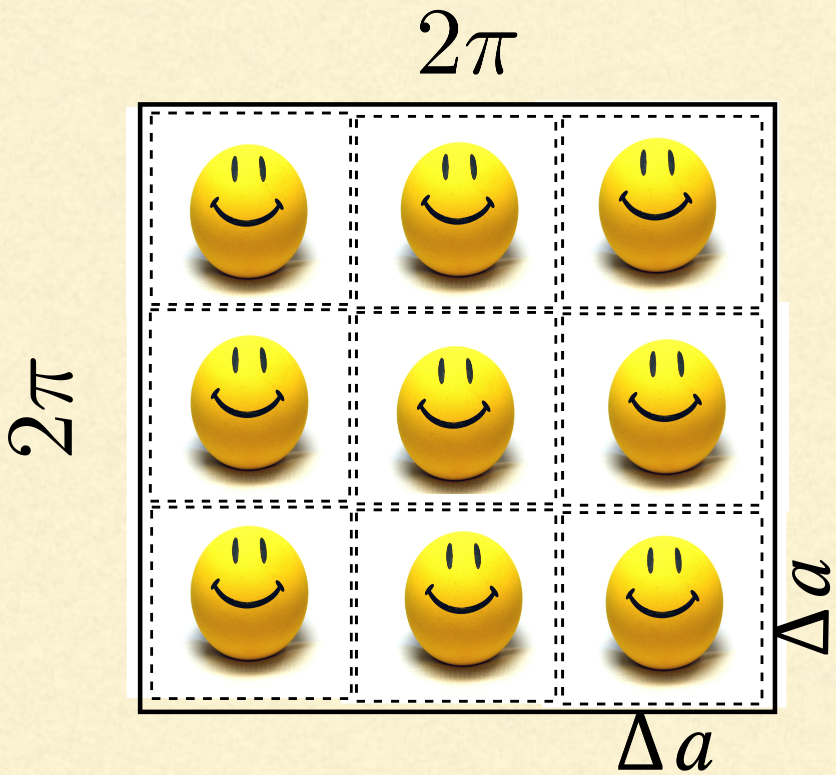
$$q(x_i, y_j, t) = \sum_k Q_k \phi\left(\frac{x_i - X_k}{r}\right) \phi\left(\frac{y_j - Y_k}{r}\right)$$

$$\Delta \psi = q - h$$

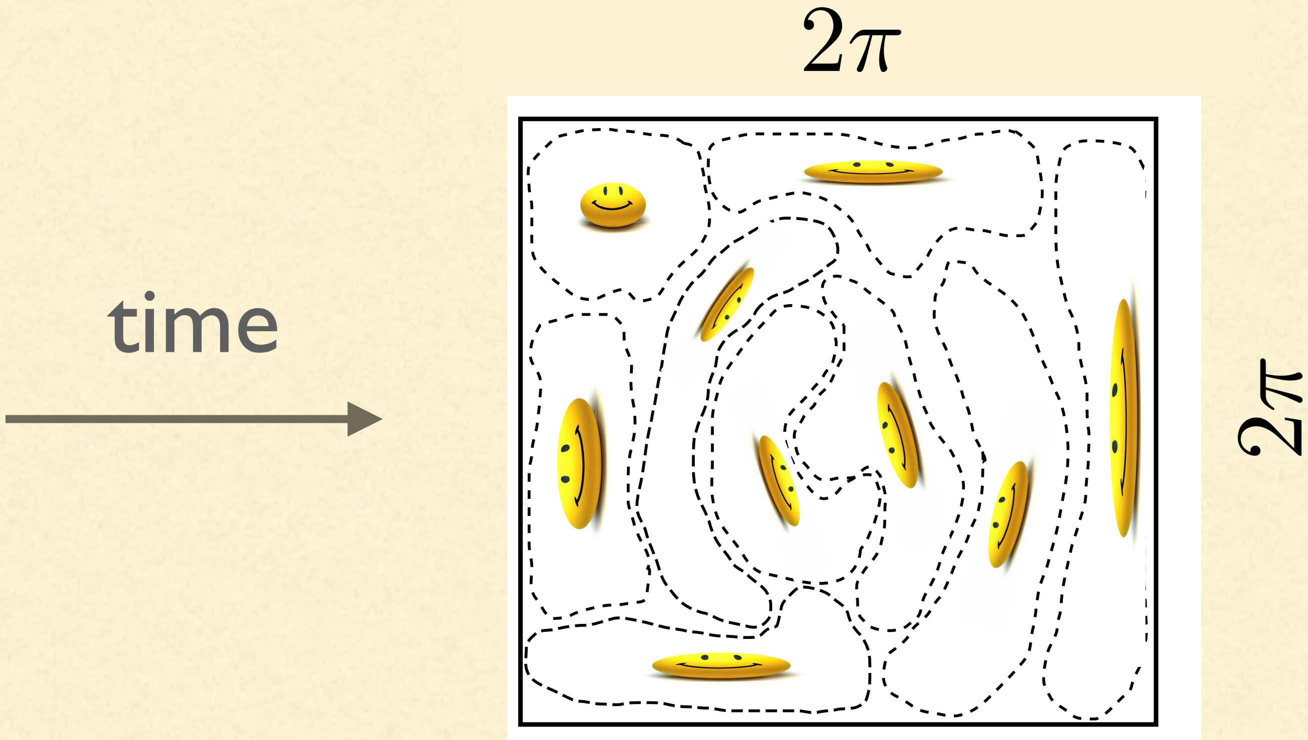
$$\frac{d}{dt} X_k = -\frac{\partial}{\partial y} \psi(x, y, t)\Big|_{(x,y)=(X_k(t), Y_k(t))}$$

$$\frac{d}{dt} Y_k = +\frac{\partial}{\partial x} \psi(x, y, t)\Big|_{(x,y)=(X_k(t), Y_k(t))}$$

AREA PRESERVATION



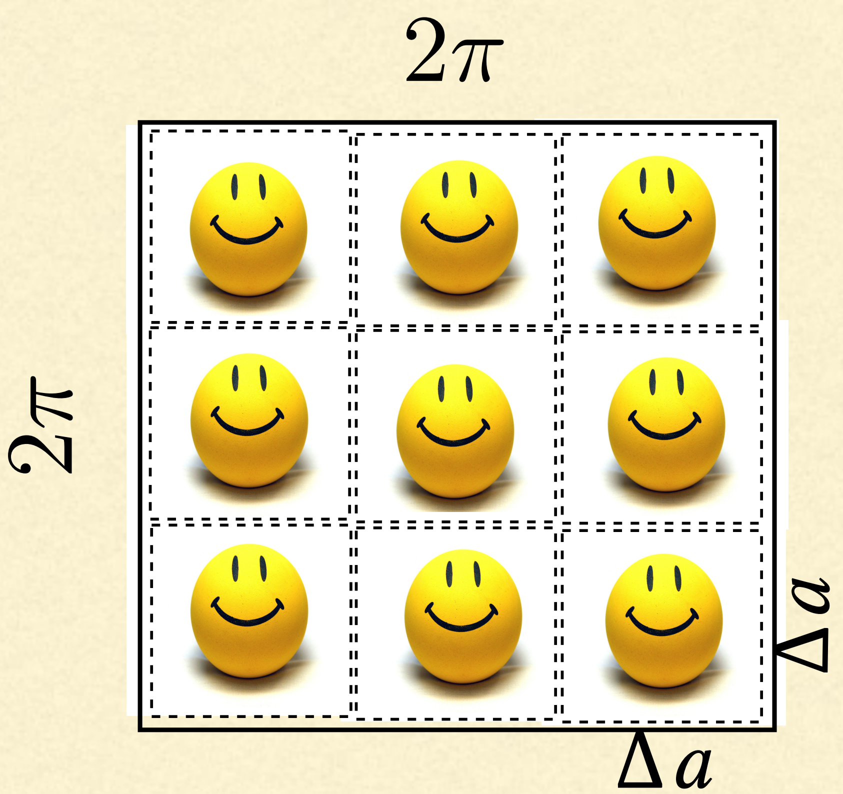
time



We initialise K particles on a uniform grid with vorticity $Q_k, \; k = 1, \dots, K$

Area associated with each Q_k is preserved over time under the divergent-free flow

LEVEL SETS OF VORTICITY



We initialise K particles on a uniform grid with vorticity $Q_k, \ k = 1, \dots, K$

Denote vorticity levels as σ_l

$\sigma_l = Q_k, \ l = 1, \dots, L, \ \text{where } L \leq K$

Meaning that we can have $Q_k = Q_{k'}$

But we can't have $\sigma_l = \sigma_{l'}$

An example could be

$$Q_1 = 1, \ Q_2 = 1, \ Q_3 = -1, \ Q_4 = 1, \ Q_5 = -1 \ (K = 5)$$

$$\sigma_1 = 1, \ \sigma_2 = -1 \ (L = 2)$$

PRIOR DISTRIBUTION

Let's denote K_l as the number of particles with vorticity level σ_l

Then the area associated with σ_l is

$$\Pi_l = \frac{K_l \Delta a^2}{(2\pi)^2}$$

This area is also preserved as it trivially follows from area-preservation of area associated with each Q_k

Note that $\sum_l \Pi_l = 1$

We take Π_l to be the prior distribution on vorticity

CONSERVATION PROPERTIES OF HPM

- Area preservation of vorticity level sets $\Pi_l = \frac{K_l \Delta a^2}{(2\pi)^2}$
- Energy conservation $E = -1/2 \int \psi(q-h)dxdy$
- Conservation of circulation (first order Casimir) $C = \int q dxdy$

STATISTICAL THEORY BASED ON PRIOR

The most probable macrostate $\rho^*(x, y, \sigma)$ is the maximum of

$$S = -\int dxdyd\sigma\,\rho\ln\frac{\rho}{\Pi}$$

subjected to satisfy conservations laws of energy and circulation

Then the most probable macrostate is

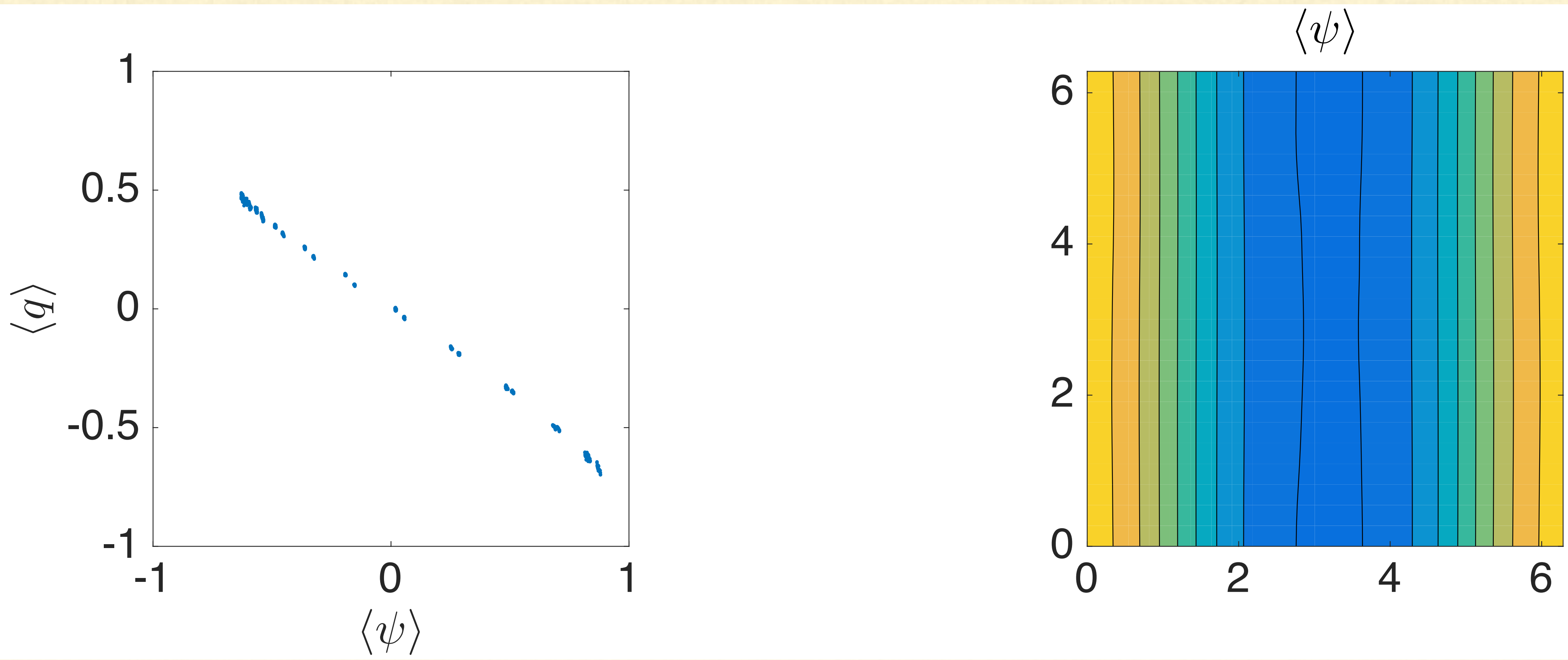
$$\rho^* = N^{-1}\exp[(-\beta\langle\psi\rangle + \alpha)\sigma]\Pi$$

The coarse-grained vorticity and stream function are

$$\langle q\rangle = \sum_l \sigma_l \rho^*(x, y, \sigma_l) \qquad \Delta\langle\psi\rangle = \langle q\rangle - h$$

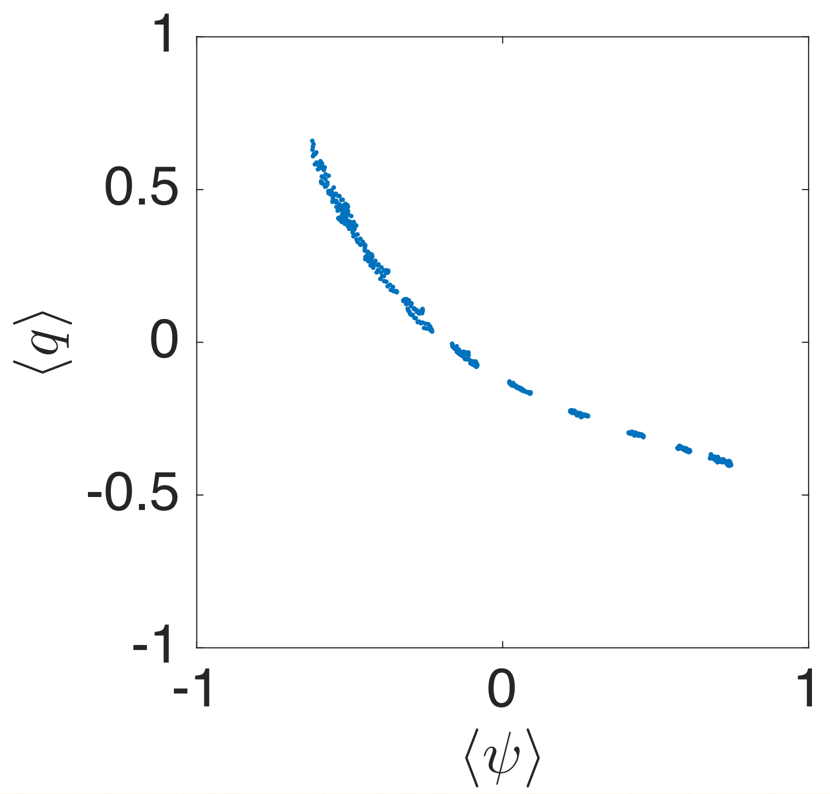
COARSE-GRAINED FIELDS OBTAINED BY THE HMP METHOD

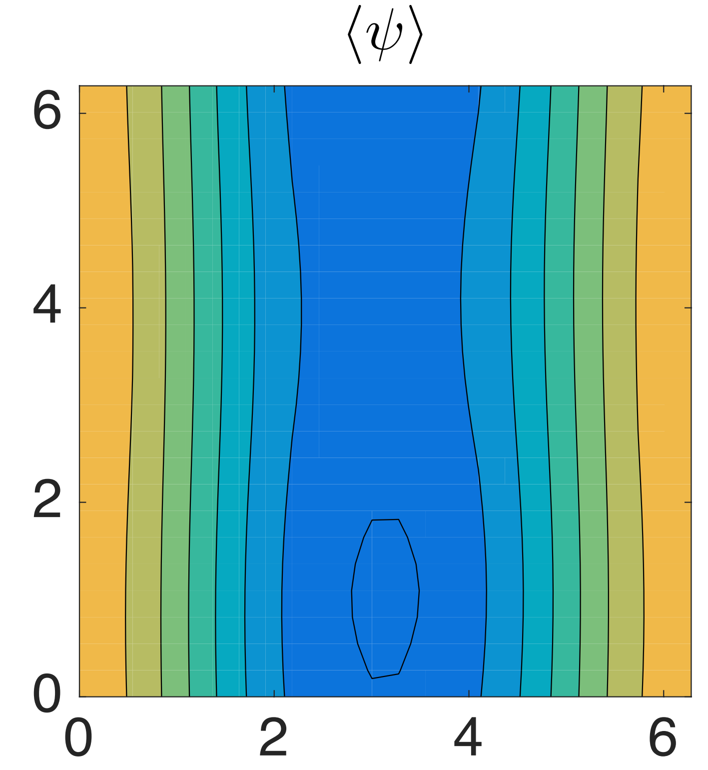
Fix energy and circulation. But change the prior.
Choose the prior Π_l as normal distribution. Then the coarse-grained fields are



COARSE-GRAINED FIELDS OBTAINED BY THE HMP METHOD

We use the same values for energy and circulation.
Choose the prior Π_l as gamma distribution. Then the coarse-grained fields are





RELATION TO DATA ASSIMILATION

- By changing the prior we can obtain different coarse-grained fields by the HPM.
- We ask the following question: can data assimilation correct model error? Meaning that if we take the HPM model for the truth but an Arakawa model for the background, can data assimilation correct the model error?
- And moreover, are the conservation properties of an Arakawa model relevant for good estimations?

EXPERIMENTAL SETUP

We derived observations from the HPM model with gamma prior distribution, since none of the Arakawa models are able to obtain such a non-linear behaviour no matter the initial conditions.

The observations were assimilated into

- Arakawa EZ model (that preserves energy and enstrophy)
- Arakawa E (that preserves energy)
- Arakawa Z (that preserves enstrophy)

As a data assimilation method we choose an Ensemble Kalman Filter with perturbed observations.

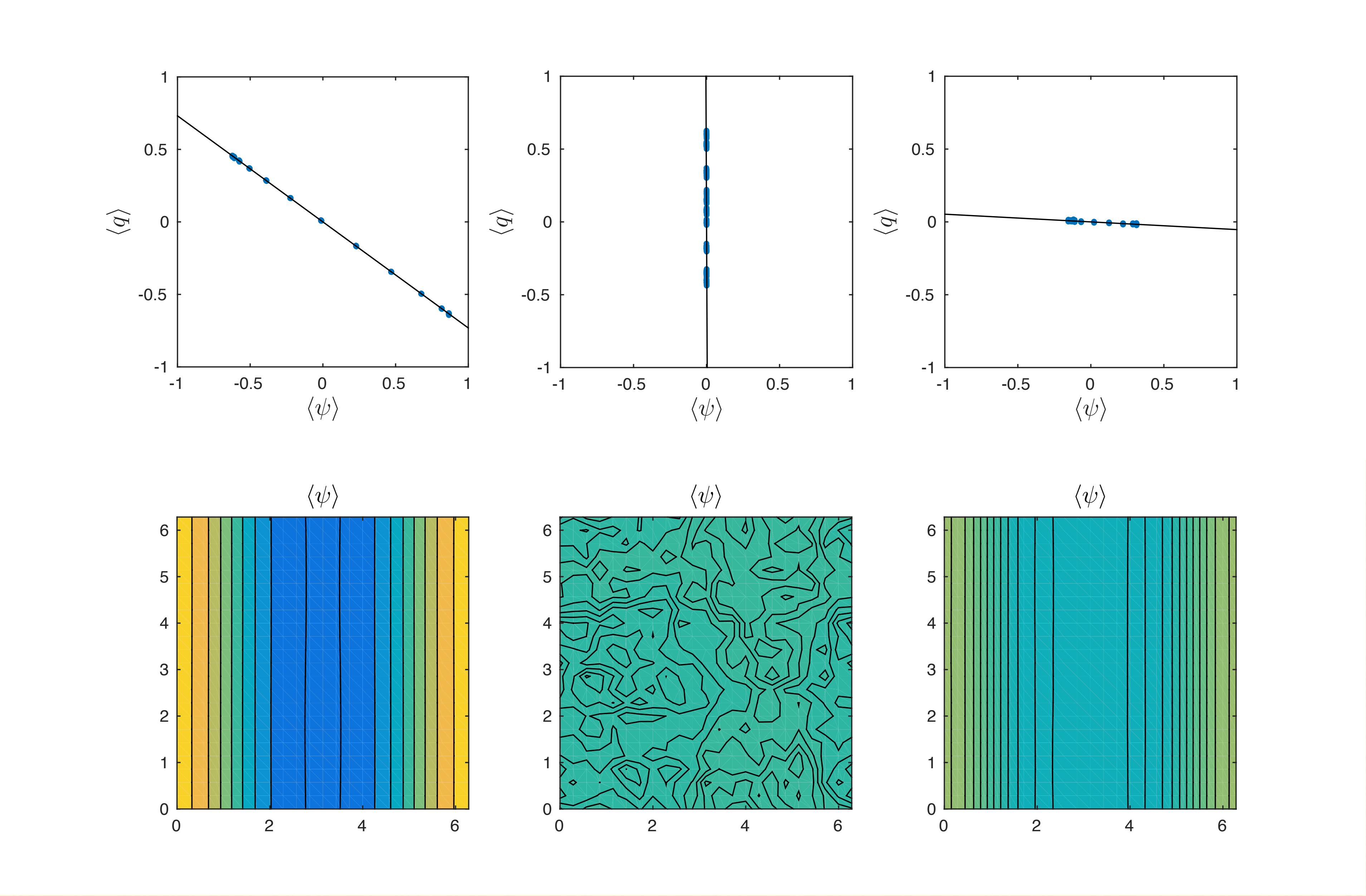
IS DATA ASSIMILATION CONSERVATIVE?

In general, data assimilation is non-conservative with a few exceptions:

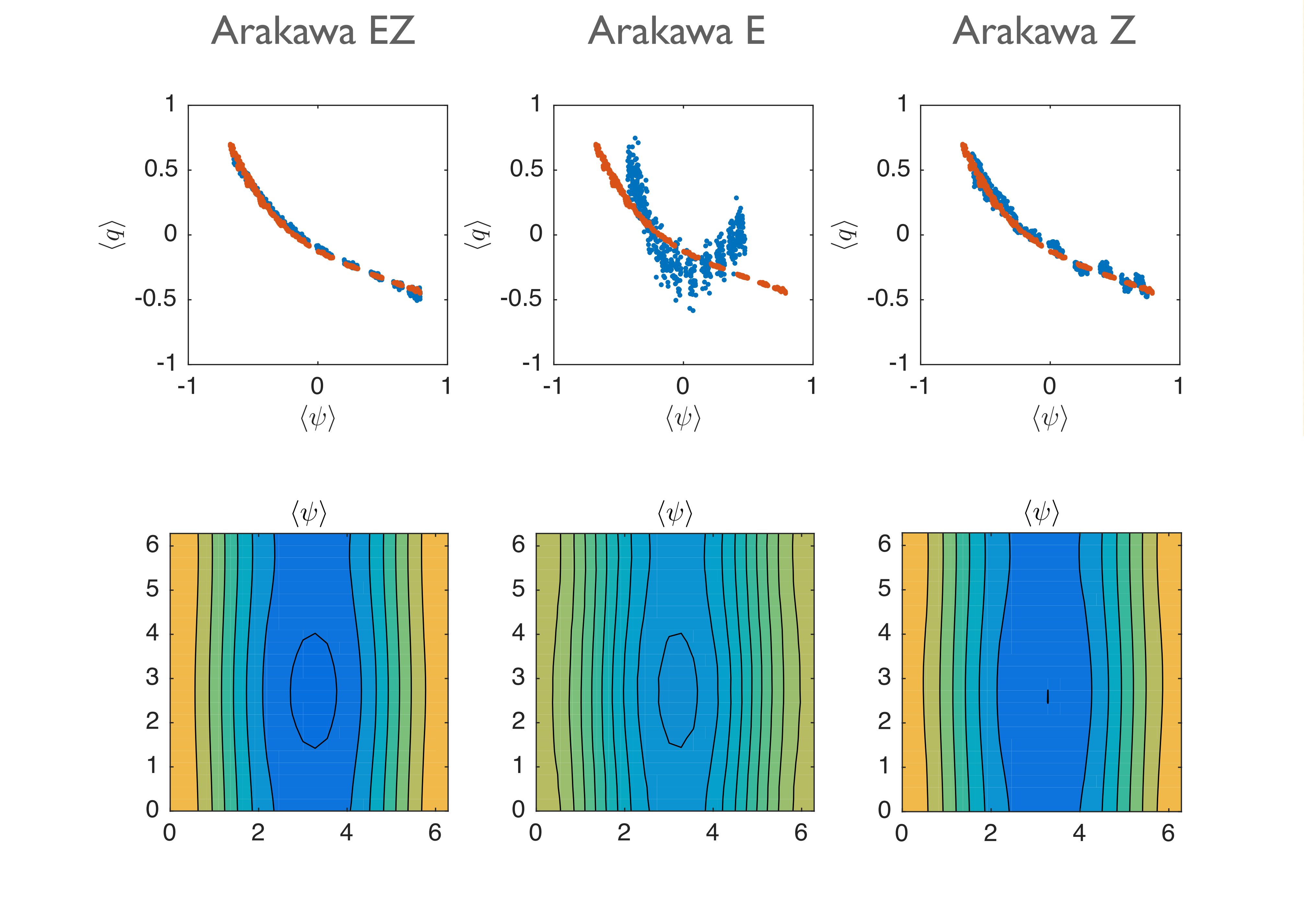
- particle fileters
- conservation of linear properties by an EnKF without localisation
- specially derived data assimilation methods that ensure conservation laws (e.g. “The Maintenance of Conservative Physical Laws within Data Assimilation Systems” by Jacobs and Ngodock, 2003; “Conservation of mass and preservation of positivity with ensemble-type kalman filter algorithms” by Janjić et al, 2014)

COARSE-GRAINED FIELDS OBTAINED BY ARAKAWA DISCRETIZATIONS

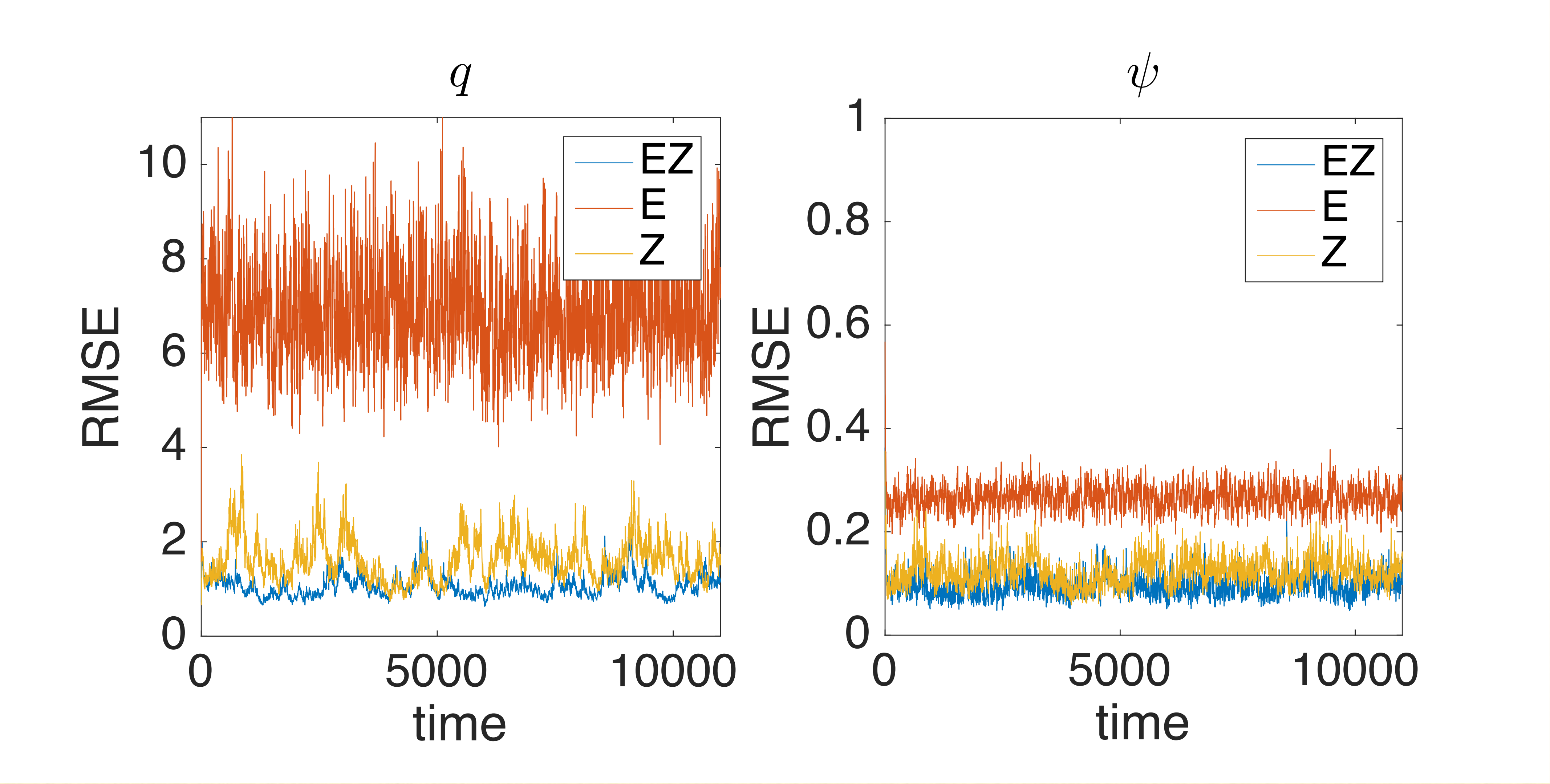
Arakawa EZ | Arakawa E | Arakawa Z



ASSIMILATION OF STREAM FUNCTION

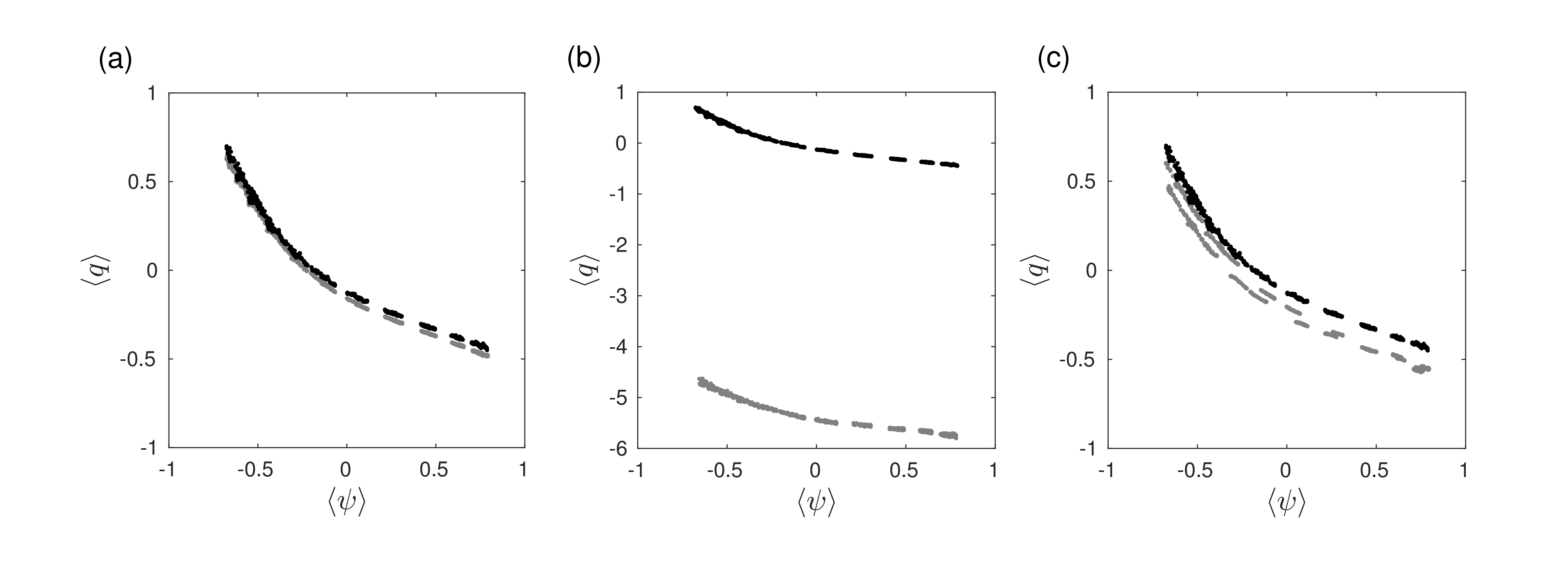


INSTANT ERRORS OBTAINED BY THE ENKF



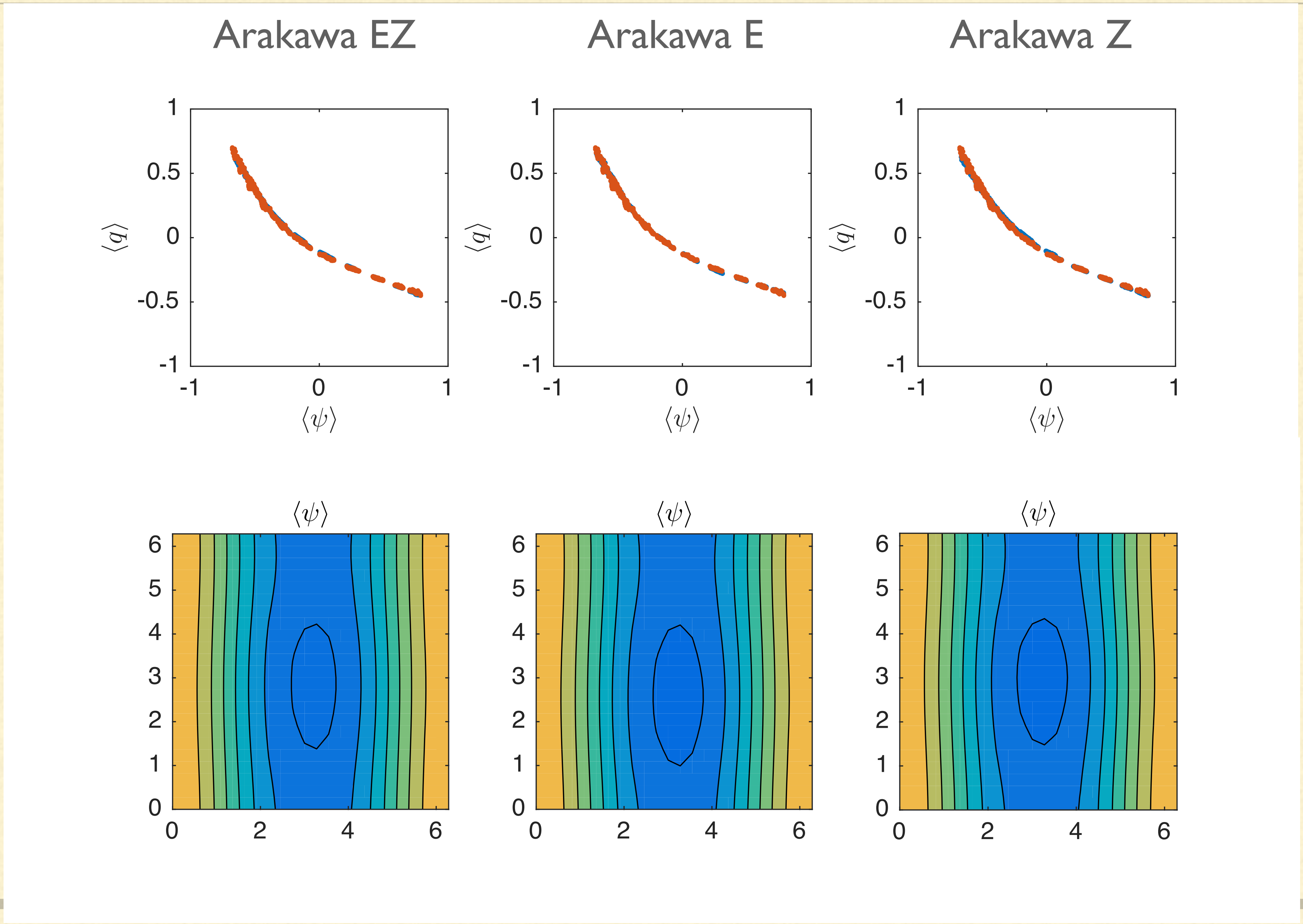
LOCALIZATION AND INFLATION

Arakawa EZ Arakawa E Arakawa Z

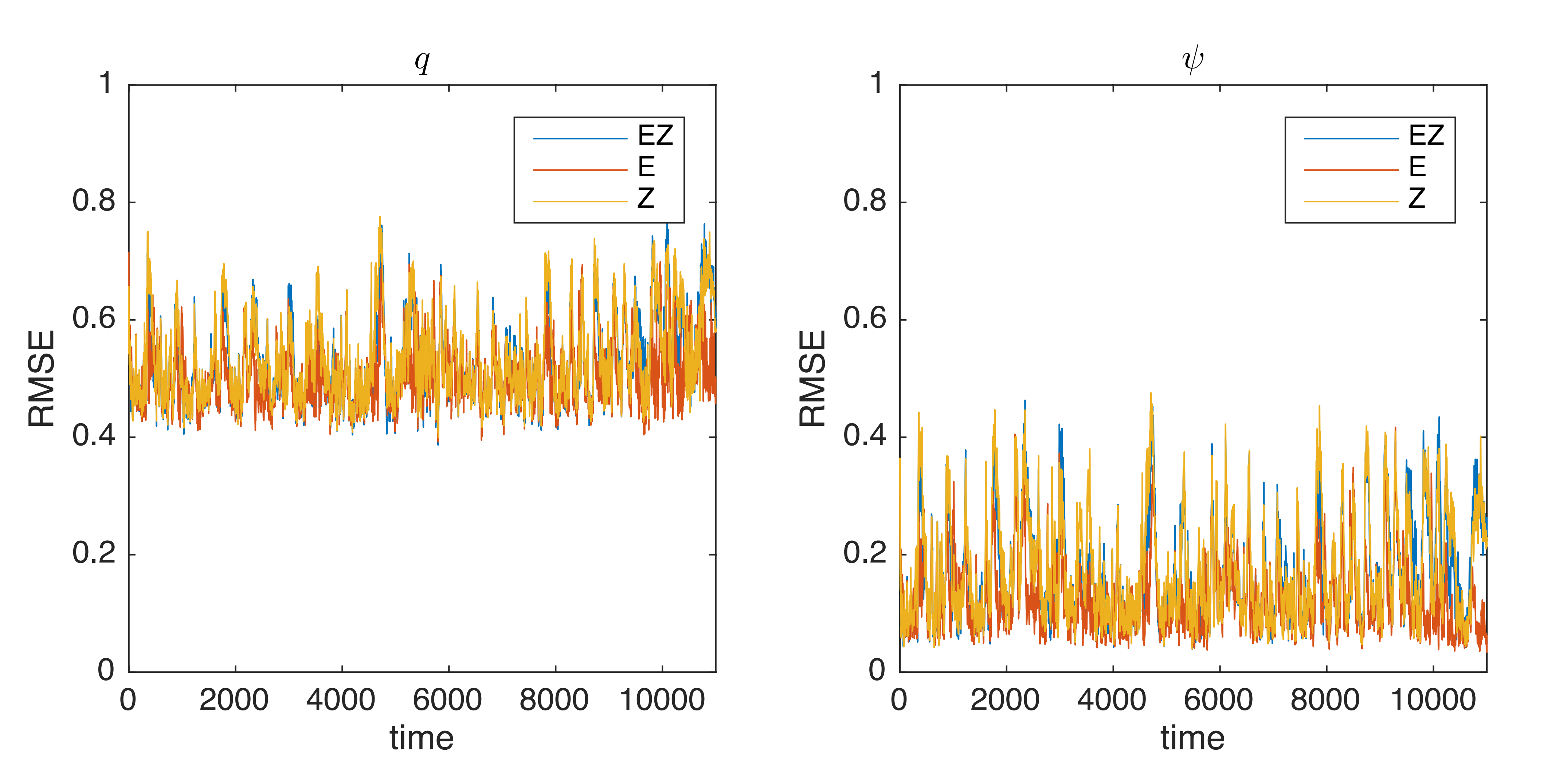


scheme	mean	std	skewness
HPM	-0.32	0.30	0.34
EZ	-0.37	0.29	0.48
E	-5.27	1.70	1.99
Z	-0.51	0.25	0.45

ASSIMILATION OF VORTICITY WITH LOCALISATION



INSTANT ERRORS OBTAINED BY THE ENKF



THE PDF OF VORTICITY

scheme	mean	std	skewness
HPM	-0.32	0.30	0.34
EZ	-0.32 -0.32	0.11 0.23	0.27 0.08
E	-0.32 -0.32	0.15 0.27	0.18 0.00
Z	-0.32 -0.32	0.10 0.23	0.27 -0.06

First column without inflation. Second with inflation.

CONCLUSIONS

- When assimilating observations of stream function the choice of a numerical model is crucial: the Arakawa EZ model that preserves both energy and enstrophy gives the best estimation. EnKF combined with Arakawa EZ estimates well the posterior mean, standard deviation and *skewness*.
- When assimilating observations of vorticity, the choice of a numerical model is not that crucial anymore. The skewness, however, estimated worse than when assimilating observations of stream function. Moreover, inflation deteriorates skewness estimation even more.